# BELYI MAPS AND DESSINS D'ENFANTS
# LECTURE 7

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## CONTENTS



## I. REVIEW

Last time we:

(1) Defined what a branch of a multi-valued function is
(2) Finished defining hyperelliptic curves
(3) Defined differentials and showed there are no nonzero holomorphic differentials on $\mathbb{P}^1$.

## II. DIFFERENTIALS, CONT.

**Example 1.** Let $E : y^2 = f(x)$ be an affine elliptic curve. We claim that the differntial $\omega = \dfrac{dx}{y}$ is holomorphic. Implicitly differentiating, we see that

$$2y\,dy = f'(x)\,dx \implies \frac{dx}{y} = 2\frac{dy}{f'(x)}\,.$$

Let $U$ be the open set where $y \neq 0$ and $V$ be the open set where $f'(x) \neq 0$. Note that these two sets cover $E$ [why?], and the above calculation shows that we can extend the definition of $\omega$ to all of $E$ by taking $\omega|_V = 2\dfrac{dy}{f'(x)}$. [Why is this meromorphic on $V$?]

**Lemma 1.** *Let $X$ be a Riemann surface, $U \subseteq X$ be a chart, and $\omega$ be a meromorphic differential on $U$. Then there is at most one meromorphic differential $\eta$ on $X$ such that $\eta|_U = \omega$.*

*Proof.* This follows from the Identity Theorem for meromorphic functions. □

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**Remark 2.**

(1) It is possible for there to be zero such extensions. For instance, consider the differential $\omega = e^z\, dz$ on $\mathbb{C}$, thought of as a chart on the Riemann sphere. There is no extension of $\omega$ to a meromorphic differential on $\widehat{\mathbb{C}}$ because $e^z$ has an essential singularity at $\infty$.

(2) The same result is true for meromorphic functions.

**Definition 3.** The genus $g(X)$ of a compact, connected Riemann surface $X$ is the dimension of the space $\Omega(X)$ of holomorphic differentials as a $\mathbb{C}$-vector space.

**Remark 4.** One can show that this agrees with the topological notion of genus.

**Proposition 5.** *Let $X$ be a compact, connected Riemann surface of genus $g$ and $\omega \in \mathcal{M}^{(1)}(X)$ be a meromorphic differential on $X$. Then the number of zeroes of $\omega$ minus the number of its poles, counted with multiplicity, is $2g-2$.*

**Remark 6.** See Abhyankar's *Historical Ramblings in Algebraic Geometry and Related Algebra* for many more characterizations of the genus of an algebraic curve.

Thus we see that differentials are useful because they allow us to define invariants of a given Riemann surface $X$. They are also useful for embedding a Riemann surface in projective space.

Let $\omega_1, \ldots, \omega_g$ be a basis of holomorphic differentials on a compact, connected Riemann surface $X$ of genus $g$. Given a point $P \in X$, write $\omega_i|_U = f_i\, dz$ on some neighborhood $U$ of $P$ with local coordinate $z$. Define

$$\begin{aligned} \kappa_U : U &\to \mathbb{P}^{g-1} \\ P &\mapsto [f_1(P) : \cdots : f_g(P)] \end{aligned}$$

and define $\kappa : X \to \mathbb{P}^{g-1}$ by $\kappa(P) = \kappa|_U(P)$ where $U$ is some chart containing $P$.

Note that this map is well-defined by the transformation property of differentials. Suppose $P \in V$ for some other neighborhood $V$ of $P$ with local coordinate $w$, and $\omega_i|_V = h_i\, dw$. Then $h_i\, dw = f_i\, dz$ on $U \cap V$, so

$$[f_1(P) : \cdots : f_g(P)] = \left[h_1(P)\frac{dw}{dz}(P) : \cdots : h_g(P)\frac{dw}{dz}(P)\right] = [h_1(P) : \cdots : h_g(P)]\,.$$

This shows that the definition of $\kappa$ is independent of the choice of chart, so $\kappa$ is well-defined.

**Definition 7.** With notation as above, the map $\kappa : X \to \mathbb{P}^{g-1}$ is called the canonical map.

- If $g = 0$, there are no nonzero holomorphic differentials.
- If $g = 1$, then $\mathbb{P}^{g-1} = \mathbb{P}^0$, which is just a point.
- If $g = 2$, then $\mathbb{P}^{g-1} = \mathbb{P}^1$. It turns out that all genus 2 curves are hyperelliptic (which we may prove later), and $\kappa$ is the degree 2 map which, when $X$ is in Weierstrass form, is simply the projection $(x, y) \mapsto x$.
- If $g \geq 3$ and $X$ is not hyperelliptic, then $\kappa$ gives an embedding of $X$ into projective space.

## III. Topological properties of surfaces and the Riemann-Hurwitz formula

### III.1. **Triangulations and Euler characteristic.**

**Definition 8.** Let $X$ be a compact topological surface.

(1) A curved triangle in $X$ is a subspace $A$ of $X$ and a homeomorphism $h : T \to A$ where $T$ is a closed triangular region in the plane. If $e$ is an edge of $T$, then $h(e)$ is an edge of $A$; if $v$ is a vertex of $T$, then $h(v)$ is a vertex of $A$.

(2) A triangulation of $X$ is a collection $\{A_i\}_i$ of curved triangles in $X$ such that $\bigcup_i A_i = X$ and for all $i \neq j$ the intersection $A_i \cap A_j$ is either
- empty;
- a vertex of both $A_i$ and $A_j$; or
- and edge of both $A_i$ and $A_j$.

(In other words, if two triangles intersect in two vertices, then they must also intersect in the edge between them.)

(3) If $X$ admits a triangulation, then it is triangulable.

**Theorem 9** (Radó). *Every topological surface is triangulable.*

**Definition 10.** Let $X$ be a compact topological surface and $T$ is a triangulation of $X$ with $v$ vertices, $e$ edges, and $f$ triangles (faces). The Euler characteristic of $X$ with respect to $T$ is

$$\chi(X) := v - e + f\,.$$

**Theorem 11.**

*(a) The Euler characteristic $\chi(X)$ is independent of the choice of triangulation, hence is an invariant of the surface $X$.*

*(b) If $X$ is an orientable, compact topological surface of genus $g$, then $\chi(X) = 2 - 2g$.*

*Proof.* This is proven in Girondo–González-Diez, Proposition 1.54. □

### III.2. **The Riemann-Hurwitz formula.**

**Theorem 12** (Riemann-Hurwitz). *Let $\pi : X \to Y$ be a morphism of compact, connected Riemann surfaces. Then*

$$2g(X) - 2 = \deg(\pi)(2g(Y) - 2) + \sum_{P\in X} (e_P(\pi) - 1)\,.$$

*Proof.* Since $X$ is compact, then the set of ramification points is finite, so the sum on the righthand side is finite.

Take a triangulation $T$ of $Y$ such that each ramification value of $\pi$ in $Y$ is a vertex of a triangle. (Given any triangulation, this condition can be satisfied: add a vertesx at each ramification value and subdivide the triangle containing it.) Let $v$, $e$, and $f$ be the number of vertices, edges, and faces of $T$. Pull $T$ back by $\pi$ to obtain a triangulation $T'$ of $X$, and denote its number of vertices, edges, and faces by $v', e'$ and $f'$. Note that every ramification point of $\pi$ is a vertex in $T'$.

Since there are no ramification values lying in the interior of a triangle in $T$, then each triangle on $Y$ lifts to $\deg(\pi)$ triangles on $X$, so $f' = \deg(\pi)f$. Similarly, $e' = \deg(\pi)e$. Fix a vertex $Q \in Y$. The number of preimages of $Q$ in $X$ is $\#\pi^{-1}(Q)$. Note that

$$\begin{aligned}\deg(\pi) &= \sum_{P\in\pi^{-1}(Q)} e_P(\pi) = \sum_{P\in\pi^{-1}(Q)} 1 + \sum_{P\in\pi^{-1}(Q)} (e_P(\pi)-1) \\ &= \#\pi^{-1}(Q) + \sum_{P\in\pi^{-1}(Q)} (e_P(\pi)-1)\end{aligned}$$

so

$$\#\pi^{-1}(Q) = \deg(\pi) - \sum_{P\in\pi^{-1}(Q)} (e_P(\pi)-1)\,.$$

Thus the total number of vertices in $X$, which is the total number of preimages of vertices in $Y$ is given by

$$\begin{aligned}v' &= \sum_{Q \text{ vertex of } Y} \#\pi^{-1}(Q) = \sum_{Q \text{ vertex of } Y} \left(\deg(\pi) - \sum_{P\in\pi^{-1}(Q)} (e_P(\pi)-1)\right) \\ &= \deg(\pi)v - \sum_{Q \text{ vertex of } Y} \sum_{P\in\pi^{-1}(Q)} (e_P(\pi)-1) \\ &= \deg(\pi)v - \sum_{P \text{ vertex of } X} (e_P(\pi)-1)\,.\end{aligned}$$

Thus

$$\begin{aligned}2g(X)-2 = -\chi(X) &= -v' + e' - f' \\ &= -\deg(\pi)v + \sum_{P \text{ vertex of } X} (e_P(\pi)-1) + \deg(\pi)e - \deg(\pi)f \\ &= -\deg(\pi)\chi(Y) + \sum_{P \text{ vertex of } X} (e_P(\pi)-1) \\ &= \deg(\pi)(2g(Y)-2) + \sum_{P\in X} (e_P(\pi)-1)\end{aligned}$$

where the last equality holds because every ramification point of $\pi$ is a vertex of $X$. □

**Remark 13.** The Riemann-Hurwitz formula is very useful for computing the genus of a Riemann surface. Often one takes $Y = \mathbb{P}^1$, so then it suffices to compute the ramification indices $e_P(\pi)$ in order to determine the genus of $X$.

**Example 14.** Let $E : y^2 = f(x)$ be an elliptic curve given by a Weierstrass equation, and let $r_1, r_2, r_3$ be the roots of $f$. Consider the projection

$$\begin{aligned}\pi : E &\to \mathbb{P}^1 \\ [X:Y:Z] &\mapsto [X:Z]\end{aligned}$$

which has degree 2. As we have previously seen, $\pi$ is ramified at $[r_j : 0 : 1]$ for $j = 1,2,3$ and at $[0:1:0]$, all with ramification index 2. By the Riemann-Hurwitz formula, we have

$$2g(E) - 2 = 2(2g(\mathbb{P}^1)-2) + \sum_{P\in E} (e_P(\pi)-1) = 2(-2)+1+1+1+1 = 0$$

confirming that $g(E) = 1$.

**Corollary 15.** *Let $\pi : X \to Y$ be a morphism of compact, connected Riemann surfaces.*

*(a) $g(X) \geq g(Y)$. In particular, if $g(X) = 0$, then $g(Y) = 0$.*
*(b) If $g(Y) = 0$ and $g(X) > 0$, then $\pi$ must be ramified.*
*(c) If $g(Y) = 1$, then $\pi$ is unramified iff $g(X) = 1$.*

## IV. THE FUNDAMENTAL GROUP AND COVERING SPACES

IV.1. **A crash course on the fundamental group.** Here we collect some useful facts about fundamental groups. For a more complete treatment, I recommend Rotman's *An Introduction to Algebraic Topology*.

**Definition 16.** Let $X$ be a topological space and $P \in X$.

- A path on $X$ is a continuous map $\gamma : [0,1] \to X$. A loop based at $P$ is a path $\gamma$ on $X$ such that $\gamma(0) = \gamma(1) = P$.
- Two loops $\gamma_1$ and $\gamma_2$ based at $P$ are (pointed) homotopic if there is a continuous map $H : [0,1] \times [0,1] \to X$ such that
$$H|_{0\times[0,1]} = \gamma_1, \qquad H|_{1\times[0,1]} = \gamma_2$$
and $H(s,0) = H(s,1) = P$ for all $s \in [0,1]$. Such an $H$ is a (pointed) homotopy of $\gamma_1$ and $\gamma_2$.

[See picture on p. 42 of Rotman.]

**Lemma 2.** *Homotopy is an equivalence relation on the set of loops based at $P$.*

Given a path $\gamma$, we denote its homotopy class by $[\gamma]$.

**Definition 17.** Given two paths $\gamma_1, \gamma_2$ with $\gamma_1(1) = P = \gamma_2(0)$, we define their concatenation by
$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & \text{if } 1/2 \leq t \leq 1\,. \end{cases}$$

In other words, $\gamma_1 * \gamma_2$ is the path $\gamma_1$ followed by the path $\gamma_2$, with the variable appropriately rescaled so that the domain is still $[0,1]$.

**Definition 18.** Let $X$ be a topological space and $P \in X$. The fundamental group of $X$ (based at $P$) is the set of homotopy classes of loops based at $P$, and is denoted $\pi_1(X,P)$. The space $X$ is simply connected if $\pi_1(X,P)$ if it is connected and $\pi_1(X,P)$ is the trivial group for some (hence, any) choice of basepoint $P \in X$.

**Proposition 19.** *The fundamental group is a group under the concatenation operation defined above.*

[What is the identity element?]

**Remark 20.** We'll abuse notation slightly as follows. If $\gamma$ is a loop on $X$, we denote by $\gamma^{-1}$ the reverse path given by
$$\gamma^{-1}(t) := \gamma(1-t)\,.$$

**Lemma 3.** *Assume $X$ is path connected and let $P, Q \in X$. Then $\pi_1(X,P) \cong \pi_1(X,Q)$.*

*Proof idea.* Let $\alpha : [0,1] \to X$ be a path from $P$ to $Q$, so $\alpha(0) = P$ and $\alpha(1) = Q$. Given a loop $\gamma$ based at $P$, then $\alpha^{-1} * \gamma * \alpha$ is a loop based at $Q$. Similarly, given a loop $\delta$ based at $Q$, we obtain a loop $\alpha * \delta * \alpha^{-1}$ based at $P$. These two maps are homomorphisms and mutually inverse up to homotopy, hence provide the desired isomorphism. □

IV.2. **Covering spaces.** Covering spaces provide a powerful tool for computing fundamental groups. They are also in some sense (which can be made very precise) the topological analogue of an algebraic closure, so studying covering spaces is like a topological version of Galois theory.

Historically, they often arose when people were trying to solve differential equations on a space that had "holes", i.e., was not simply connected. Often the problem couldn't be solved on the starting space, but did have a solution after passing to a suitable cover.

**Definition 21.** Let $X$ be a topological space. A covering space of $X$ is a topological space $E$ together with a continuous map $\pi : E \to X$ called a covering map such that the following property holds. For each $P \in X$ there exists a neighborhood $V$ of $P$ such that $\pi^{-1}(V) = \bigsqcup_i U_i$, where the sets $U_i$ are pairwise disjoint and the restriction $\pi|_{U_i} \to V$ is a homeomorphism. We say that such a neighborhood $V$ is evenly covered by $\pi$.

**Example 22.** Let $X = \mathbb{S}^1 \subseteq \mathbb{C}$ be the circle, considered as the set of points $z$ with $|z| = 1$. Then

$$\begin{aligned} \pi : X &\to X \\ z &\mapsto z^2 \end{aligned}$$

is a covering space.

**Example 23.** Consider again the circle $X = \mathbb{S}^1 \subseteq \mathbb{C}$. Then

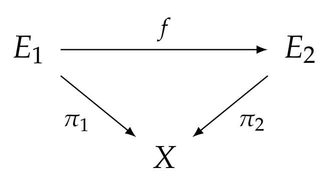
$$\begin{aligned} \pi : \mathbb{R} &\to \mathbb{S}^1 \\ t &\mapsto e^{2\pi i t} \end{aligned}$$

is a covering space of $X$. (One can visualize $\mathbb{R}$ embedded in $\mathbb{R}^3$ as a helix with $p$ the projection map down to the plane.) [See picture on p. 49 of GGD.]

**Remark 24.** The fibers $\pi^{-1}(P)$ of a covering are discrete.

**Remark 25.** If $X$ is a Riemann surface, then $E$ inherits a unique holomorphic structure such that the covering map $\pi : E \to X$ is holomorphic. The idea is that we simply pull back the charts of $X$ to $E$: given a chart $(U, \varphi)$ on $X$, define a chart $(\pi^{-1}(U), \varphi \circ \pi)$ on $E$.

**Definition 26.** Let $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ be coverings of $X$. A morphism from $\pi_1$ to $\pi_2$ is a continuous map $f : E_1 \to E_2$ such that $\pi_1 = \pi_2 \circ f$, i.e., such that the following diagram commutes.




The map $f$ is an isomorphism of coverings if it is a homeomorphism.

**Definition 27.** A deck transformation of a covering $\pi : E \to X$ is an automorphism of the covering. The set of deck transformations of $\pi$ is a group, denoted $\mathrm{Deck}(E/X)$ or $\mathrm{Deck}\left(E \xrightarrow{\pi} X\right)$.

**Theorem 28.** *Let $X$ be a connected topological space. Then there exists a covering $\pi : \widetilde{X} \to X$ with $\widetilde{X}$ connected and simply connected. Moreover $\widetilde{X}$ is unique up to isomorphism.*

**Definition 29.** The covering space $\widetilde{X}$ in the previous theorem is called the universal covering space of $X$.

**Example 30.**

- The covering map $\pi : \mathbb{R} \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$ is the universal cover since $\mathbb{R}$ is simply connected.
- Let $\Lambda \subseteq \mathbb{C}$ be a lattice and $X = \mathbb{C}/\Lambda$ be the corresponding torus. Then the quotient map $\pi : \mathbb{C} \to X$ is the universal cover, since $\mathbb{C}$ is simply connected.

Covering spaces possess some important lifting properties.

**Lemma 4** (Path-lifting lemma)**.** *Let $\pi : E \to X$ be a covering space. Let $\gamma$ be a path on $X$ and let $P = \gamma(0)$. Given any preimage $e \in \pi^{-1}(P)$ there exists a unique path $\widetilde{\gamma}$ on $E$ such that $\pi \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = e$.*

**Definition 31.** Such a $\widetilde{\gamma}$ is called a lift of $\gamma$ based at $e$.

**Lemma 5** (Uniqueness of lifts)**.** *Let $\pi : E \to X$ be a covering space of $X$, let $Y$ be a connected topological space, and let $f : (Y, y_0) \to (X, x_0)$ be a continuous map of pointed spaces. (This just means that $f(y_0) = x_0$.) Given $e_0 \in \pi^{-1}(x_0)$ there exists at most one continuous map $\widetilde{f} : (Y, y_0) \to (E, e_0)$ with $\pi \circ \widetilde{f} = f$, i.e., such that the following diagram commutes.*